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Why Do We Use the Dutch Auction to Sell Flowers?
—Information Disclosure in Sequential Auctions

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Abstract

This paper characterizes equilibria in various two-stage sequential auction formats under all possible forms of interperiod information disclosure in an IPV model. We study the role of interperiod information disclosure in affecting bidders' intertemporal learning, bidding, and auction revenue. Unlike Milgrom and Weber (1982), who show in their model that it is always good for the auctioneer to commit to complete information revelation, we find that this is not necessarily the case in sequential auctions due to bidders' intertemporal substitution. In our model, only selective information release can be revenue enhancing. We show that the standard sequential Dutch auction or the sequential first-price auction with the announcement of each stage's winning bid generates the highest revenue among all considered auction formats.

JEL classification: C72, D44, D82.

Keywords: Sequential auction, Information disclosure, Intertemporal substitution, Revenue comparison.

1 Introduction

Many nondurable goods auctions are carried out repeatedly across periods, such as flower auctions and fish auctions, etc. It is interesting to observe that these goods are often sold via the Dutch auction format or its variants. The sale of flowers is a well known example for its use of the Dutch auction (about 85% of Netherlands cut flowers are handled by the Dutch auctions annually[1]). The fish auction is a natural variant of the Dutch format. Here then comes the puzzle. In the auction literature, we know that if bidder are risk-neutral and their valuations are independent and identically distributed, the first-price, second-price, Dutch and English auctions are revenue equivalent; and once their valuations are affiliated, the English auction generates the highest revenue followed by the second-price auction and then the Dutch and first-price auctions. In both cases, the Dutch auction never beats the other auction procedures in term of revenue. But why do people stick to it in various nondurable goods auctions? In this paper, we provide one explanation, that is, the Dutch auction can be revenue superior in a sequential environment that captures the essential features of the nondurable goods sale.[2]

There are four important features for most nondurable goods auctions. First, those goods are sold period by period because they are nondurable and the goods sold each period are approximately the same. Second, bidders' identities are the same across periods, and these bidders tend to be large buyers in the same line of business aiming for the retail resale of the auctioning goods. Third, because of the similarity of each period's goods, the valuations of the same bidder in two consecutive rounds tend to be correlated. Finally, due to this valuation correlation and

[1]The figure is quoted from the International Labor Organization working paper *The World Cut Flower Industry: Trends and Prospects*.

[2]Notice that in practice flowers are sold repeatedly via a kind of multi-unit Dutch auction. Because of the analytical difficulties, this paper only models the sequential single-unit auction.

constant bidder identity, each bidder can always infer some information on her rivals' current valuations from the released bidding results of previous rounds. This interperiod learning makes bidding behavior very different from that in a one-shot auction.

In this paper, we analyze the sequential sale via a highly stylized independent private value (IPV) model, where two bidders compete for two identical nondurable objects, each per period. In connection with the flower sale, the private value can be interpreted as the private gross profit (before deducting the bid) of each buyer in the industry. Then the goal of the auctioneer is to select an auction format with appropriate interperiod information release to maximize her overall revenue. We assume that the auctioneer will commit to one auction format for both periods. The commonly adopted formats include: the first-price and second-price auctions, the English and Dutch auctions. The choices of information disclosure at the end of each stage auction include: announcing winning or losing status, winning or losing bids or both.[3] This in turn defines 16 sequential auction formats, most of which are outcome equivalent as we will discuss later. The objective of this paper is to characterize the Perfect Baysian Nash Equilibrium (PBNE) in various sequential auction formats and then carry out the revenue comparison to find out the optimal one.

The distinction of this paper from other works on sequential auctions is briefly summarized as follows. It differs from McAfee and Vincent (1997) in the following respect. McAfee and Vincent (1997) deal with the sale of a single object, which will be resold if it can not be auctioned when all bids in a given period are below the reserve price. This paper, however, studies the sequential sale of two objects, each of which can always be auctioned at each period since we will assume no reserve price. The difference between this paper and Weber (1983) is that Weber

[3]Notice that the *minimum* information released in each period is the winning or losing status because bidders need to know their entitlement to the object, which is nondurable hence must be consumed in the current period.

considers a sequential auction with unit-demand and a bidder quits the auction once she obtains one unit, while this paper studies a sequential sale where a bidder pursues a unit every period.

There are three contributions in this paper. First, we characterize the equilibria in various two-stage sequential auctions with multi-unit demand. Second, we analyze the effect of interperiod information release on bidding behavior and the revenue. Finally, from the revenue comparison of all sequential formats, we obtain the result that the standard Dutch auction is tied with the first-price auction with the winning-bid announcement, generating the highest revenue. Considering the implementation simplicity of the standard Dutch format relative to its first-price counterpart, this result may explain the pervasive employment of Dutch auctions in flowers and fish sale.

The remainder of the paper proceeds as follows. Section 2 describes the model. Equilibrium bidding behavior will be characterized in Section 3. Section 4 carries out revenue comparison. Section 5 concludes.

2 The Model

This is a two-period sequential auction model with two risk-neutral bidders. The auctioneer has two identical nondurable goods for sale, one per period. The auction formats we consider include: sequential first-price, second-price, English and Dutch auctions with the announcement of winning and losing status, of winning bid, of losing bid and of both bids, at the end of the first stage. Bidder i's valuation is v_i, where $i = 1, 2$. Both valuations are iid over the unit interval [0, 1] according to the density $f(v)$, which is continuously differentiable and bounded away from zero. After each bidder observes her valuation at the first round, her valuation remains constant since then, i.e., her second-stage valuation will be the same as her first-stage one. So here we assume a perfect cross-period correlation of valuations. The discount factor is δ, where $\delta \in [0,1]$. Each bidder will maximize her discounted sum of surplus and the auctioneer her discounted sum of revenue.

3 Characterization of Equilibrium

Since we assume bidders are ex ante identical, it is natural for us to focus on symmetric monotonic PBNE. As we will see later, in some formats, bidders will have to adopt a mixed strategy. In this paper, we are going to restrict ourselves to those equilibria with a monotonicity requirement defined as follows. Let $I(v)$ and $S(v)$ denote the inf and sup of a bidder's randomized bids with valuation v at a given stage.[4] Whenever $v_2 > v_1$, we should have $I(v_2) > I(v_1)$ and $S(v_2) > S(v_1)$. Notice that the cost of this monotonicity requirement is that it may rule out pooling equilibria and may give rise to a non-existence problem, which is indeed the case as we see later. The benefit of it, however, is to ensure the revenues are compared within the same category of equilibria (monotonic equilibria only) so that we can maintain the maximum uniformity when discussing the ranking here relative to the other rankings in literature that typically use monotonic equilibria.

3.1 Outcome Equivalence Simplification

In the sequential IPV environment, the standard Dutch auction has the same equilibrium outcome as the first-price auction with the announcement of winning bid at each stage. This is because when a standard Dutch auction, e.g., the flower auction, ends, all bidders can publicly observe the winning bid. Of course, the standard Dutch auction technology can also be altered

[4] A pure strategy is considered a degenerate mixed strategy.

to accommodate all other information disclosure requirements. These variants of the standard Dutch auction are outcome equivalent to their sequential first-price counterparts too. So there is no loss of generality for us to focus our subsequent analysis on the sequential first-price auctions only.

Similarly, in the sequential environment, the standard English auction has the same equilibrium outcome as the second-price auction with the announcement of losing bid at each stage. That is because when a standard English auction ends, all bidders can publicly observe all the losing bids. Also, the standard English auction technology can be modified to meet all other information release requirements. Again, because of the outcome equivalence, our analysis can only be concentrated on the sequential second-price auctions.

Therefore, we will study the sequential first-price and second-price auctions with the announcement of winning or losing status, winning bid, losing bid and both bids in the rest of the paper.

3.2 Sequential Second-Price Auctions

The bidding behavior in the sequential second-price auctions is easy to characterize because bidders bid the same way regardless of interperiod information release structures. The following proposition states the equilibrium strategy.

Proposition 1 *In the two-stage sequential second-price auctions with perfect cross-period correlation of valuations, bidding one's own valuation at both stages constitutes a monotonic equilibrium.*

Proof. See the Appendix.

However here bidding one's own valuation will not constitute a dominant strategy equilibrium any more as in the one-shot second-price auction. Also, it is interesting to observe that different interperiod information disclosures do not affect bidders' bidding behavior at all. This is because bidding one's own valuation is still a dominant strategy for the last stage, which gives bidders no incentive to deviate from the equilibrium at the first stage. It is not obvious whether there exist other symmetric monotonic equilibria or not. The current equilibrium will be the only one if we assume bidders simply want to play a stage dominant strategy at each period.

3.3 Sequential First-Price Auctions

The bidding behavior in the sequential first-price auctions is much more complicated. Under different interperiod information release structures, bidders bid differently. Consequently the interperiod information disclosure plays a crucial role for the auctioneer's revenue. We begin with the analysis of information release of the first-stage winning/losing status, where the identity of winner or loser will be announced once the first-stage auction ends.[5]

3.3.1 Announcement of the First-Stage Winning/Losing Status

We are looking for a monotonic pure strategy symmetric equilibrium in this game and it has the following structure.

Equilibrium beliefs and strategies:

1. At the first stage, each bidder bids according to the bid function $\beta(v)$.
2. At the second stage, if the bidder wins the first stage at the valuation v, she believes that

[5]Notice that announcing the winner's or the loser's identity discloses the same information in the two-bidder model.

her rival's valuation $\hat{v}$ falls in the interval of $[0, v)$ with the conditional density $\frac{f(\hat{v})}{F(v)}$.[6] Then she bids according to $\beta_1(v)$. While if she loses the first stage, she believes that her rival's valuation $\hat{v}$ will be in $[v, 1]$ with the conditional density $\frac{f(\hat{v})}{1-F(v)}$, and bids according to $\beta_2(v)$.

Notice that at the second stage each bidder' belief on her rival's valuation distribution is parameterized by her own private signal. However, in a standard one-shot auction environment, bidders' beliefs on valuation distribution are common knowledge. This difference makes it impossible to directly apply the standard equilibrium existence results in the auction literature to here. But similarity between the structure of our setting and the standard auction environment enables us to extend established approaches to the current case. The following proposition gives us a confirmation of the existence of the above equilibrium in the sequential first-price auction.

Proposition 2 *In the two-stage sequential first-price auction with perfect cross-period correlation of valuations, there exists a monotonic pure-strategy symmetric equilibrium under the first-stage winning/losing status announcement.*

Proof. See the Appendix.

The proof makes use of the results in Landsberger, Rubinstein, Wolfstetter and Zamir (2001), who study an auction environment with commonly known ranking of valuations. It is exactly our second-stage problem. Theorem 1 in Landsberger et al (2001) gives us an equilibrium existence and uniqueness result among all differentiable bid functions for the second-stage problem. Using this result, the existence of the first-stage equilibrium bid function is immediate. In fact, we can further show that both stages' equilibrium bid functions must be differentiable drawing on the method in Maskin and Riley (2003), which combined with Theorem 1 in Lands-

[6]As we are looking for a symmetric monotonic equilibrium, a bidder's winning implies her rival's valuation is smaller than hers.

berger et al (2001) yields the uniqueness of pure-strategy equilibrium for this two-stage problem. As we will not focus on the uniqueness issue in this paper, its proof is forgone.

The symmetric monotonic pure-strategy equilibrium can not be analytically solved. This makes a final revenue comparison impossible. But a revenue ranking is important to illuminate the role of information disclosure in sequential auctions. So we assume the valuation density $f(v) = 1$, i.e., we assume bidders' valuations are iid uniform over $[0,1]$. Our subsequent revenue ranking will be under this simplifying assumption. In the current case the closed-form solutions to the bid functions still can not be obtained under the iid uniform assumption. However, we can find some qualitative features for the equilibrium bid functions, which are collected in the following proposition and sufficient for our final revenue comparison.

Proposition 3 *Under the assumption that bidders' valuations are iid uniform, the equilibrium bid functions have the following properties:*

a) $\beta_1(0) = \beta_2(0) = 0$ *and* $\beta_1(1) = \beta_2(1) = t^*$, *where* t^* *is both bidders' common terminal bid.*

b) $\beta_1(v)$ *and* $\beta_2(v)$ *are strictly concave and convex respectively.*

c) $\frac{3}{4}v > \beta_2(v) > t^*v > \beta_1(v) > \frac{1}{2}v$ *for all* $v \in (0,1)$.

d) $\frac{2}{3} \geq t^* \geq \frac{5}{8}$

e) $\beta(v) = \frac{v}{2}(1-\delta) + \frac{\delta}{v}\int_0^v \beta_2(t)\,dt < \frac{v}{2}$ *for all* $v \in (0,1)$.

Proof. See the Appendix.

Property a) says that two bid functions at the second stage has the same starting and ending points. This result is standard. Property b) shows that both bid functions behave regularly, which is a result due to the uniform distribution assumption. Property c) offers us some bounds to approximate the two bid functions. $\frac{v}{2}$ and $\frac{3v}{4}$ can be shown to be tangent to $\beta_1(v)$ and $\beta_2(v)$ at the point $v = 0$ respectively. So the two bid functions are enclosed by their respective tangent lines at the origin and separated by the line t^*v. Property d) gives us rather narrow bounds for the common end bid t^*. Property e) gives us an expression for the first-stage bid function, which is smaller than $\frac{v}{2}$ for all $v \in (0,1)$, where $\frac{v}{2}$ is just the equilibrium bid function in a one-shot auction of two bidders with iid uniform signal. Also from Property e), we can see that the first-stage bid function is an increasing function of δ.

These properties will be useful for our revenue comparison. Also they give us a very intuitive understanding of bidders' behavior in this sequential first-price auction. First, because of information disclosure, bidders have less private information at the second stage than in a standard one-shot auction. So they can obtain less informational rent, which explains why both of them bid more aggressively in the second round than in a one-shot auction. Second, after the first-stage auction, the first-period loser will believe that her rival is stronger than her previous expectation while the winner will believe her rival is weaker than her previous expectation. This induces the loser to bid more aggressively than the winner at the second stage. Third, both bidders will bid less aggressively than in a one-shot auction in the first round, which we call *bid reduction* in this paper. Bid reduction is a direct result of bidders' optimal decision of intertemporal substitution. In the second stage, the first stage winner and loser bid quite differently, which provides an intertemporal arbitrage opportunity for bidders. In the first stage, it is profitable for a bidder to bid less than in a one-shot auction only. This is because by doing so at the second stage this bidder will have a higher chance to meet a first-stage winner, who is easier to defeat than a first-stage loser. This explains why in equilibrium a bid reduction can occur in the first stage. Finally, as to the effect of the discount factor δ, once it becomes bigger, the second-stage payoff has higher value for the bidder, which naturally promotes her intertemporal substitution, i.e., leading to larger first-stage bid reduction.

3.3.2 Announcement of the First-Stage Winning Bid

Now, we come to the analysis of bidding behavior under more interperiod information disclosure—announcing the winning bid at the end of the first stage. So not only the winning and losing status, but also the winning bid becomes common knowledge at the second stage. Again, we are looking for a symmetric monotonic equilibrium. It is easy to show that there is no pure-strategy monotonic equilibrium in this game. So we focus our attention to the following equilibrium, where both bidders follow the same monotonic pure-strategy bid function in the first stage and the first-stage winner will adopt a mixed strategy at the second stage.

Equilibrium beliefs and strategies:

1. At the first stage, each bidder bids according to the bid function $\beta(v)$.

2. At the second stage, if the bidder wins the first stage with her valuation v, then she believes that her rival's valuation falls in the interval of $[0, v)$ with the conditional density $\frac{f(\hat{v})}{F(v)}$. She will randomly choose a bid b on the support $(t_*, t^*]$ with density $g^v(b)$.[7] While if a bidder loses the first stage at the valuation v and infers that the winning valuation is $\tilde{v}$ from the winning bid announcement, she will bid $\beta_2^{\tilde{v}}(v)$.[8]

There are two important features of the above equilibrium. One is that the first-stage winner adopts a randomized strategy at the second stage. The other is that both bidders have to condition their second-stage bids on the announcement of the first-stage winning bid. The equilibrium existence result for general density $f(.)$ can be shown in the similar backward induction manner as in the Proof of Proposition 2. Notice that the second-stage equilibrium is a generalization of the asymmetric auction example in Vickrey (1961). So by applying the refinement argument in

[7] Notice that for different v, the randomization density g is also different.

[8] Also the functional form of $\beta_2^{\tilde{v}}(.)$ is parameterized by the announcement.

Vickery (1961), we will have a unique equilibrium outcome here with an additional assumption that bidders will always choose strategies involving least mixing. For the purpose of revenue comparison, we will derive the specific equilibrium strategies only for the uniformly distributed valuations.

Proposition 4 *Under the assumption that bidders' valuations are iid uniform, in the two-stage sequential first-price auction with perfect cross-period correlation of valuations, bidders will exhibit the following equilibrium behavior under the first-stage winning bid announcement: both bidders bid* $\frac{v}{2}$ *in the first stage; the first-stage winner randomizes over* $(\frac{v}{2}, \frac{3v}{4}]$ *according to the c.d.f.* $G^v(b) = \frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}$*; the first-stage loser will not bid if her valuation* $v \in [0, \frac{\tilde{v}}{4})$*, where* $\tilde{v}$ *is the inferred valuation of the first-stage winner, and bid* $\tilde{v} - \frac{\tilde{v}^2}{4v}$ *if* $v \in [\frac{\tilde{v}}{4}, \tilde{v})$.

Proof. See the Appendix.

The above is a symmetric monotonic equilibrium as we define it at the beginning of Section 3. Notice that if the first-stage winner's valuation is $\tilde{v}$ and the first-stage loser's valuation is v, the winner will bid above $\frac{\tilde{v}}{2}$ and the loser will bid $\tilde{v} - \frac{\tilde{v}^2}{4v}$. $\tilde{v} - \frac{\tilde{v}^2}{4v}$ is smaller than $\frac{\tilde{v}}{2}$ for all $v < \frac{\tilde{v}}{2}$. This means that in equilibrium when the first-stage loser's valuation is smaller than a half of the winner's, the loser always loses the second stage. Under this contingency, other bid functions for the first-stage loser can also constitute a monotonic equilibrium as long as the submitted bid is smaller than a half of the winner's valuation and at the same time prevents the winner's deviation. Of course, these equilibria will all yield the same outcome. But our equilibrium in Proposition 4 is the only one that can describe the first-stage loser's strategy in just one function, hence making the derivation of the first-stage bid function tractable.[9]

The intuition for the above equilibrium bidding behavior is straight forward. First, since the first-stage winner's valuation is always commonly known at the beginning of the second stage, it is not surprising that the winner will randomize in order to offset this information asymmetry. Second, the first-stage loser obtains some informational advantage at the second stage, i.e., knowing the winner's valuation. As the loser is weaker than the winner in the first place, this extra information enables both bidders to compete on a relatively *level* ground. So in equilibrium no bidder's second-stage bid function can dominate the other. In contrast, in the previous case the loser's bid function always lies above that of the winner. Third, there is no bid reduction in the first stage because there exists no intertemporal arbitrage opportunity. At the second stage, it is as hard to defeat a first stage loser as to defeat a winner since now two bidders' strength is brought in line with each other by the loser's informational advantage. So no bidder has the incentive to overbid or underbid in the first stage. Finally, the discount factor δ does not enter the first-stage bid function. This is natural since bidders do not need to consider the intertemporal substitution at all.

3.3.3 Announcement of the First-Stage Losing Bid

Now instead of announcing the winning bid, the auctioneer can also choose to only release the first-stage losing bid in a sequential first-price auction. Before we start to solve for the equilibrium, our intuition from the previous analysis will lead to the following conjecture. The

[9] Kaplan and Zamir (2000) derive almost the same second-stage equilibrium in their Proposition 5.2. The only difference is that they assume that the first-stage loser bids her own valuation when her valuation is less than a half of the winner's inferred valuation. The equilibrium in Kaplan and Zamir (2000) truncates the bid function $\beta_2^{\tilde{v}}(v)$, which makes the analytical derivation of the first-stage bid function rather difficult.

first-stage loser is already on the weak side at the second stage. Announcing her bid gives further informational advantage to the winner, which only exacerbates the loser's position and will induce quite aggressive bidding for the loser at the second stage. Consequently there will be a large bid reduction in the first stage. This is because by following the same intertemporal arbitrage reasoning, the first-stage bidder will have a high incentive to underbid so that her chance to meet an aggressive first-stage loser can be decreased. Although the above conjecture is in the right direction, the following proposition shows that the strength of the first-stage bid reduction can be so big that no symmetric monotonic equilibrium can be supported regardless of the value of discount factor δ.

Proposition 5 *In the two-stage sequential first-price auction with perfect cross-period correlation of valuations, there is no symmetric monotonic equilibrium under the first-stage losing bid announcement.*

Proof. See the Appendix.

It is not clear if there exist other asymmetric or non-monotonic equilibria in this case. But the non-existence of an important class of equilibria here may shed light on the phenomenon that it is very rare to observe any real world sequential first-price or Dutch auction arrangements where the auctioneer discloses each stage's losing bid only.

3.3.4 Announcement of both the First-Stage Winning and Losing Bids

Under this format, the auctioneer releases all the information available to her, i.e., both the winning and losing bids, by the end of the first stage. If we assume a symmetric monotonic pure-strategy bid function for the first stage, then both bidders will exactly infer each other's valuation from the interperiod information release. At the second stage, if a bidder loses the first stage with a valuation v, then she will face a rival whose valuation is $\hat{v}$ where $\hat{v} > v$. The type of the second-stage equilibria we are looking for is similar to those discussed in Blume (2003). The loser will randomize according to the density $h(v)$ over the support $[v' - \eta, v')$ where $v' \in [v, \hat{v})$, $\eta > 0$ and the winner with high valuation $\hat{v}$ bids v'. So the sup of the loser's randomized bids should be between her and the winner's valuations and the whole randomization builds a wall that prevents the winner from bidding less than this sup. We assume that bidders choose strategies with least mixing. This assumption will lead to a large class of equilibria parameterized by v', η, and the randomization density $h(v)$. It is analytically intractable to derive those equilibria when the parameters remain general. So we will only focus on a subset of this class of equilibria. As we will see below, this subset of equilibria turn out to generate the same auction revenue.

Since $v' \in [v,\ \hat{v})$, it is natural to set v' as a weight average of v and $\hat{v}$, i.e., $(1-k)v + k\hat{v}$, where $k \in (0,\ 1)$. For analytical convenience, we set η as $v' - v$, i.e., we simply let the first-stage loser randomize over the support of $[v,\ (1-k)\,v + k\hat{v})$. We also assume $h(v)$ to be a uniform density. So the equilibrium will be as follows:

Equilibrium beliefs and strategies:

1. At the first stage, each bidder bids according to the bid function $\beta(v)$.

2. At the second stage, both the first-stage winner's and loser's valuations $\hat{v}$ and v become common knowledge. The loser randomizes uniformly over the support $[v,\ (1-k)\,v + k\hat{v})$, while the winner bids $(1-k)\,v + k\hat{v}$.

The above equilibrium is a symmetric and monotonic one according to our definition. But in order to support it, we need an extra assumption specified in the following lemma.

Lemma 1 *The above equilibrium is supportable only when* $\frac{1}{2} \geq k \geq \frac{\delta}{1+\delta}$.

Proof. See the Appendix.

When $k > \frac{1}{2}$, the first-stage loser's randomization density wall is not high enough to prevent the winner's penetration (deviation) at the second stage. While if $k < \frac{\delta}{1+\delta}$, a bidder can always profitably mimic the zero valuation at the first stage. This deviation can only be prevented by asking the first-stage loser to bid sufficiently above her valuation so as to stop the winner from bidding too leniently at the second stage, which in turn will eliminate bidders' incentive to underbid in the first stage. The final equilibrium is summarized in the following proposition.

Proposition 6 *Under the assumption that bidders' valuations are iid uniform, in the two-stage sequential first-price auctions with perfect cross-period correlation of valuations, bidders will exhibit the following behavior under both the first-stage winning and losing bids announcement: both bidders bid* $\frac{(1-\delta k)v}{2}$ *in the first stage, where* $k \in [\frac{\delta}{1+\delta}, \frac{1}{2}]$. *The first-stage loser randomizes uniformly over* $[v_l, (1-k)v_l + kv_h)$ *and the first-stage winner bids* $(1-k)v_l + kv_h$, *where* v_l *and* v_h *are the realized valuations of the loser and the winner respectively.*

Proof. See the Appendix.

The above result has almost the same interpretation as in the case of sequential first-price auctions with the announcement of winning/losing status. The first-stage loser bids aggressively at the second stage. Bid reduction appears in the first stage, which becomes more serious when discount factor δ gets bigger. We can also check that the total revenue in this two-stage auction is $\frac{1}{3} + \frac{1}{3}\delta$ (see Lemma 4 in next section), which does not contain the weight k. It means that bidders' intertemporal substitution exactly cancels out the effect of k. So the revenue remain the same within this subset of equilibria. We conjecture that this property may be extended to the original large class of equilibria.

4 Revenue Comparison

By now, we have either derived equilibria or their properties in all considered sequential auction formats under the iid uniform assumption. We are ready to compare the revenue generated from each of them. Due to the simplification at the beginning of Section 3, we only need to consider the following four revenues. First, the revenue from the sequential second-price auctions, which is denoted as R_1. Second, the revenues from the sequential first-price auctions with the announcement of the first-stage winning/losing status, winning bid, and both winning and losing bids. We denote them as R_2, R_3, R_4 respectively. Notice that there is no symmetric monotonic equilibrium in the sequential first-price auction with the announcement of the first-stage losing bid, so we leave it out of our revenue comparison.

From Proposition 1, we know that $R_1 = \frac{1}{3} + \frac{\delta}{3}$. The calculations for the other revenues are much more involved. Lemma 2 gives us an upper bound for R_2 although no bid functions can be analytically obtained for the case.

Lemma 2 *The revenue* $R_2 < \frac{1}{3} + \frac{16}{49}\delta$.

Proof. See the Appendix.

The bound of R_3 is stated in Lemma 3.

Lemma 3 *The revenue* $R_3 > \frac{1}{3} + \frac{1}{3}\delta$.

Proof. See the Appendix.

Finally, we need to calculate R_4. Lemma 4 states the result.

Lemma 4 *The revenue* $R_4 = \frac{1}{3} + \frac{1}{3}\delta$.

Proof. See the Appendix.

Proposition 7 *Under the assumption that bidders valuations are iid uniform, the revenue ranking is:* $R_3 > R_1 = R_4 > R_2$.

The following table summarizes all the revenue results we have obtained. We use I, II and III to denote R_3, R_1 and R_2 respectively. So I, II and III represent revenues in descending order. The first row of the table represents 4 different information release structures, where from left to right more and more information is disclosed. The first column represents four basic stage auction rules, where the first-price and second-price are outcome equivalent to the Dutch and English auctions respectively.

Table I. Revenue from Sequential Auctions with iid Uniform Valuations

Auction Formats	W/L Status	W Bid	L Bid	W & L Bids
First/Dutch	III	I	—	II
Second/English	II	II	II	II

From the above table, we can easily see that the interperiod information disclosure is immaterial in the sequential second-price and English auctions because the intertemporal learning does not affect bidders' equilibrium bidding. Notice that in sequential environment, the first-price and Dutch formats can revenue dominate the second-price and English formats. This is because under the first two the auctioneer has an extra device, i.e., the intertemporal information disclosure, to affect bidders' bidding and increase the total revenue.

It is commonly known that more information revelation from the auctioneer can further facilitate bidders' competition, hence increasing the revenue. A series of theorems in Milgrom and Weber (1982), which we abbreviate as MW hereafter, show that the public reporting policy never decreases the revenue in all auction formats. But the above table tells us that more information does not necessarily increase the revenue. From the announcement of the winning/losing status to the announcement of the winning bid only, revenue from the sequential first-price auctions increases. But as more information is released, i.e., both the winning and losing bids are announced, the revenue drops again.

Then we might ask why our results are so different from those in MW and how to explain the relationship between the intertemporal information disclosure and the auction revenue in our environment. There are two critical differences between our environment and that in MW. First, MW considers one-shot auction, while this paper studies sequential auctions. Second, in MW, the auctioneer's information affects both bidders' own valuations and their inference of their rivals' valuations. So the information disclosure creates both the valuation-increasing effect because bidders' valuations are assumed to be monotonically increasing with the announced signal, and the inference effect. But in this paper, the auctioneer's information does not change bidders' own valuations and we only have the inference effect here. In our setting, it is still true that more information disclosure never decreases the *stage* auction revenue. Drawing on the results in the proofs of Lemma 2 to 4, it is easy to check that the *second-stage* revenues in the first-price and Dutch formats are consistently improved by more and more intertemporal information release. However, as auctions are conducted sequentially, increased second-stage revenue does not guarantee a higher overall two-stage revenue because of bidders' intertemporal substitution. If the interperiod information structure is such that an intertemporal arbitrage opportunity exists for bidders to profitably shade their bids in the first stage, then in equilibrium a first-stage bid reduction will occur which in turn will lead to a decreased first-stage revenue. So overall, the two-stage revenue may not be enhanced. Therefore, the best information release structure should be the one that not only intensifies the second-stage bidding competition but also eliminates the intertemporal arbitrage opportunity, hence the first-stage bid reduction. The announcement of the first-stage winning bid in the sequential first-price and Dutch auctions just satisfies this informational requirement hence yielding the highest revenue among all considered formats. Then we conclude that it is not the information volume but the information structure that actually matters in term of revenue maximization in sequential auctions. The general rule is to give the auction loser some informational advantage. If we push the above argument further, it is natural to ask whether there exists such an information structure that provides an opposite intertemporal arbitrage incentive to induce overbidding in the first stage. This is possible only when the first-stage loser bids less aggressively than the winner at the second stage, which may happen under other valuation evolution assumptions but not in our setting.

5 Conclusion

This paper characterizes equilibria in various two-stage sequential auction formats under all possible forms of interperiod information release in an IPV model. We study the role of interperiod information disclosure in affecting bidders' intertemporal learning, bidding, and auction revenue. Unlike Milgrom and Weber (1982), who show in their model that it is always good for the auctioneer to commit to complete information revelation, we find that this is not necessarily the case in sequential auctions. Information disclosure does not affect the revenue in the sequential second-price and English auctions. In the sequential first-price and Dutch auctions, more information release can even decrease the revenue. This is because in sequential environment bidders' intertemporal substitution may lead to bid reduction in the first stage, which outweighs the second-stage revenue gain from the interperiod information release, hence decreasing the overall revenue. We show that the standard sequential Dutch auction or the first-price auction with the first-stage winning bid announcement generates the highest revenue among all considered formats just because their information release structure facilitates the second-stage competition and at the same time avoids the first-stage bid reduction. As a first step study of sequential auctions with multi-unit demand, our results are derived in a two-stage two-bidder model. Extensions to arbitrary number of bidders and stages; more general distribution and interperiod relation of bidders' valuations, etc., are left for future work.

Appendix: Proofs

Proof of Proposition 1.

Given bidder 1 follows and believes her rival also follows the equilibrium strategy, if bidder 2 mimics the valuation other than her own, she does not gain in the first round. In the second round, bidder 1 still bids her valuation, which is optimal even though the inferred valuation from bidder 2 is wrong because bidding one's own valuation in a single second-price auction is an ex post equilibrium. So the optimal response for bidder 2 at the second stage is to still bid her own valuation. Then there is nothing to gain for bidder 2 to mimic the other valuation at the first stage, which leads to the conclusion that in equilibrium both bidders will bid their own valuations. Q.E.D.

Proof of Proposition 2.

The existence proof consists of two standard steps. The first step produces the equilibrium candidate and the second step verifies that the candidate is indeed an equilibrium.

Step 1. Producing the equilibrium candidate. We start from the second stage. Let us assume that a bidder observes a valuation v and submit a bid $\beta(z)$ at the first stage, i.e., she mimics z type, while the other bidder follows the specified strategy truthfully. Let $\lambda(\beta)$, $\lambda_1(\beta)$ and $\lambda_2(\beta)$ denote the inverse functions of $\beta(v)$, $\beta_1(v)$ and $\beta_2(v)$ respectively.

If the bidder loses at the bid $\beta(z)$, then she bids $\beta_2(m)$, i.e., she mimics type m at the second stage, and her rival will bid $\beta_1(\hat{v})$. The losing bidder believes that $\hat{v}$ is in $(z,1]$ with density $\frac{f(\hat{v})}{1-F(z)}$. So the second-stage expected payoff for the first-stage losing bidder who mimics type m is $\pi_2 = (v-\beta_2(m))\Pr(\beta_2(m) > \beta_1(\hat{v}))$. $\Pr(\beta_2(m) > \beta_1(\hat{v}))$ is the probability of winning at the second stage for the first-stage losing bidder. In our case, $\Pr(\beta_2(m) > \beta_1(\hat{v})) =$

$\frac{F[\beta_1^{-1}(\beta_2(m))]-F(z)}{1-F(z)}$. In a standard auction environment, this probability is only a function of the current stage bid, while here the probability is also parameterized by her previous stage bid.

If the bidder wins at the bid $\beta(z)$, then she bids $\beta_1(n)$, i.e., she mimics type n, and her rival will bid $\beta_2(\hat{v})$, where the winning bidder believes that $\hat{v}$ is in $[0,z)$ with density $\frac{f(\hat{v})}{F(z)}$. So the second-stage expected payoff for the first-stage winning bidder who mimics type n is $\pi_1 = (v-\beta_1(n))\Pr(\beta_1(n) > \beta_2(\hat{v}))$. $\Pr(\beta_1(n) > \beta_2(\hat{v}))$ is the probability of winning at the second stage for the first-stage winning bidder. Here $\Pr(\beta_1(n) > \beta_2(\hat{v})) = \frac{F[\beta_2^{-1}(\beta_1(n))]}{F(z)}$.

We then start to consider the first-period bid function. The first-period bid function has to balance the second-period payoff. Since we have assumed that the bidder mimics type z at the first stage, her overall expected payoff for two stages is:

$$\pi(v) = F(z)[(v-\beta(z))+\delta\pi_1]+(1-F(z))[\delta\pi_2]$$
$$= F(z)\{(v-\beta(z))+\delta[(v-\beta_1(n))\frac{F[\beta_2^{-1}(\beta_1(n))]}{F(z)}]\}+(1-F(z))[\delta(v-\beta_2(m))\frac{F[\beta_1^{-1}(\beta_2(m))]-F(z)}{1-F(z)}]$$
$$= F(z)(v-\beta(z))+\delta(v-\beta_1(n))F[\beta_2^{-1}(\beta_1(n))]+\delta(v-\beta_2(m))[F[\beta_1^{-1}(\beta_2(m))]-F(z)]$$

The optimality of the symmetric bid functions requires that truthful bidding is optimal for all three bid functions simultaneously. Then we can use the following conventional method:

The first-order condition of z is:

(1) $f(z)v - f(z)\beta(z) - F(z)\beta'(z) - \delta f(z)(v-\beta_2(m)) = 0$

The first-order condition of m is:

(2) $(v-\beta_2(m))\beta_1^{-1'}(\beta_2(m))\, f[\beta_1^{-1}(\beta_2(m))] - \{F[\beta_1^{-1}(\beta_2(m))]-F(z)\} = 0$

The first-order condition of n is:

(3) $(v-\beta_1(n))\beta_2^{-1'}(\beta_1(n))\, f[\beta_2^{-1}(\beta_1(n))] - F[\beta_2^{-1}(\beta_1(n))] = 0$

Since the truthful bidding is the equilibrium solution to the above three equations, we replace m, n and z with v. Then (2) and (3) become:

(4) $(v-\beta_2(v))\beta_1^{-1'}(\beta_2(v))\,f[\beta_1^{-1}(\beta_2(v))]-\{F[\beta_1^{-1}(\beta_2(v))]-F(v)\}=0$

(5) $(v-\beta_1(v))\beta_2^{-1'}(\beta_1(v))\,f[\beta_2^{-1}(\beta_1(v))]-F[\beta_2^{-1}(\beta_1(v))]=0$

Let $\lambda(\beta)$, $\lambda_1(\beta)$ and $\lambda_2(\beta)$ denote the inverse functions of $\beta(v)$, $\beta_1(v)$ and $\beta_2(v)$ respectively. The equations (4) and (5) can be transformed into:

(6) $(\lambda_2(t)-t)\lambda_1'(t)\,f(\lambda_1(t))=F[\lambda_1(t)]-F[\lambda_2(t)]$

(7) $(\lambda_1(t)-t)\lambda_2'(t)\,f(\lambda_2(t))=F[\lambda_2(t)]$

The equilibrium boundary conditions must be $\lambda_1(0)=\lambda_2(0)=0$ and $\lambda_1(1)=\lambda_2(1)=t^*$ as usual, where t^* is the common terminal bid when a bidder observes the valuation 1. Theorem 1 in Landsberger et al (2001) gives the existence result of a *monotonic* solution to the system (6) and (7). The first-stage bid function can be directly solved from the equation transformed from equation (1) by imposing the equilibrium condition, i.e., replacing z and m with v. Let y and y' denote $\beta(v)$ and $\beta'(v)$ respectively. Then the transformed equation can be rewritten as: $y'=-\frac{f(v)}{F(v)}y+\frac{f(v)v}{F(v)}-\frac{\delta f(v)v}{F(v)}+\frac{\delta f(v)}{F(v)}\beta_2(v)$. This is a nonhomogeneous first-order linear differential equation. Given the boundary condition $\beta(0)=0$, the unique solution is: $\frac{1}{F(v)}\int_0^v[f(t)t-\delta f(t)t+\delta f(t)\beta_2(t)]dt$. It is easy to check that $\beta'(v)=\frac{f(v)\int_0^v F(t)(1-\delta+\beta_2'(t))dt}{F^2(v)}$. Given the fact that $\beta_2'(t)>0$ as we have obtained above, $\beta'(v)>0$ is immediate.

Step 2. Verifying the equilibrium. Since the three bid functions produced above are all monotonic, the application of the standard verification approach, i.e., to show that all other mimicking types will lead to less payoff, is straight forward. So we forgo its detailed derivation here. Q.E.D.

Proof of Proposition 3.

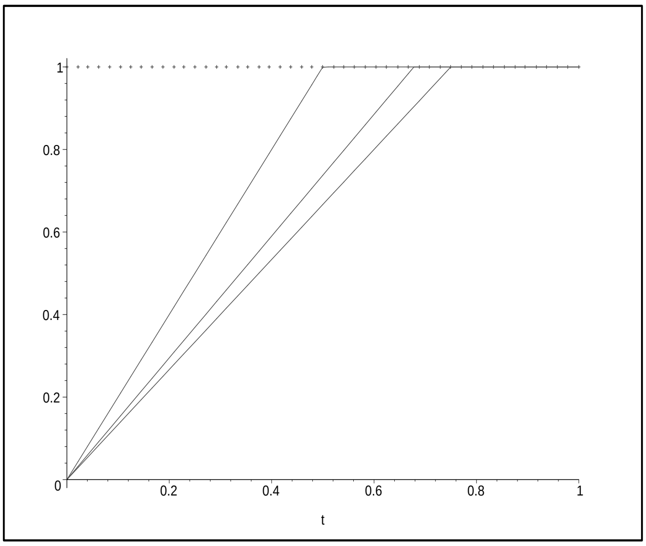
Under the assumption that $f(v) = 1$, the differential equation system (6) and (7) can be transformed into:

(8) $(\lambda_2(t) - t)\lambda_1'(t) = \lambda_1(t) - \lambda_2(t)$

(9) $(\lambda_1(t) - t)\lambda_2'(t) = \lambda_2(t)$

Property a) is just the standard boundary conditions. The following graph helps us to see the proof for the rest of the properties.

Figure 1. Bounds of Inverse Bid Functions



In the above figure, from the left to the right, the three lines are $2t$, $\frac{t}{t^*}$ and $\frac{4t}{3}$ respectively. We need to show that $\lambda_1(t)$ and $\lambda_2(t)$ behave regularly within the regions between $2t$ and $\frac{t}{t^*}$, $\frac{t}{t^*}$ and $\frac{4t}{3}$ respectively. Property b) and c) are proved by ruling out all other possibilities by contradiction, which is a tedious process. First, we can easily show that $\lambda_1'(0) = 2$ and $\lambda_2'(0) = \frac{4}{3}$. Second, we can show that $\lambda_1'(t^*) = 0$ and $\lambda_2'(t^*) = \frac{1}{1-t^*}$. Third, differentiate both sides of equation (9), we have $\lambda_1'(t)\lambda_2'(t) + (\lambda_1(t) - t)\lambda_2''(t) = 2\lambda_2'(t)$, i.e., $\lambda_2''(t) = \frac{2\lambda_2'(t) - \lambda_1'(t)\lambda_2'(t)}{\lambda_1(t) - t}$. As $\lambda_1(t) - t > 0$, we can obtain the following relations: (i) If $\lambda_1'(t) < 2$, then $\lambda_2''(t) > 0$. (ii) If $\lambda_1'(t) > 2$, then $\lambda_2''(t) < 0$. The above results prepare us to show the bounds for $\lambda_1(t)$ and $\lambda_2(t)$

with the following steps.

A. Suppose $\lambda_1(t)$ lies entirely above the line $2t$. Notice that $\lambda_1'(0) = 2$ and $\lambda_1'(t^*) = 0$, which combined with the fact that $\lambda_1(t)$ lies entirely above $2v$ implies that $\lambda_1'(t)$ must have first increased above two and then decreased to zero. Given the smoothness of $\lambda_1(t)$, this means that there exists at least a $t > 0$ such that $\lambda_1'(t) = 2$. Let $\bar{t}$ be the inf of the set of such t . Then $\lambda_1'(\bar{t}) > 2$ for all $t < \bar{t}$. By relation (ii), we must have $\lambda_2''(\bar{t}) = 0$ and $\lambda_2''(t) < 0$ for all $t < \bar{t}$. Then from equation (8), we can obtain the equation $3\lambda_2(\bar{t}) = \lambda_1(\bar{t}) + 2\bar{t}$. From equation (9), we have the equation $(\lambda_1(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \lambda_2(\bar{t})$. Combining these two equations, we obtain $3(\lambda_2(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \lambda_2(\bar{t})$. Notice that $\lambda_2'(\bar{t}) < \frac{4}{3}$ because $\lambda_2'(0) = \frac{4}{3}$ and $\lambda_2''(t) < 0$ for all $t < \bar{t}$. Then $\lambda_2(\bar{t}) < \frac{4}{3} \times 3(\lambda_2(\bar{t}) - \bar{t})$. So we must have $\frac{\lambda_2(\bar{t})}{\bar{t}} > \frac{4}{3}$, which is a contradiction to the fact that until $\bar{t}$, $\lambda_2(t)$ still lies below the line $\frac{4}{3}t$. So $\lambda_1(t)$ can not lie entirely above $2t$.

B. Suppose $\lambda_1(t)$ crosses $2t$ from above first. Suppose the crossing happens at the point $\hat{t}$, where $\hat{t} > 0$ and $\lambda_1'(\hat{t}) < 2$. Again there exists a $\bar{t}$ such that $0 < \bar{t} < \hat{t}$ and $\lambda_1''(\bar{t}) = 2$ and $\lambda_1'(\bar{t}) > 2$ for all $t < \bar{t}$. The above argument can be applied in exactly the same way here. So we can rule out this case too.

C. Suppose $\lambda_1(t)$ crosses $2t$ from below first at $\bar{t} > 0$. Then $\lambda_1'(\bar{t}) > 2$ and $\lambda_1(\bar{t}) = 2\bar{t}$. From the equation (8), we have $(\lambda_2(\bar{t}) - \bar{t})\lambda_1'(\bar{t}) = 2\bar{t} - \lambda_2(\bar{t})$. So we must have $\frac{\lambda_2(\bar{t})}{\bar{t}} < \frac{4}{3}$, i.e., $\lambda_2(t)$ goes below $\frac{4}{3}t$ at the point $\bar{t}$. As $\lambda_1'(t)$ must first decrease below 2, $\lambda_2'(t)$ will first increase above $\frac{4}{3}$ from relation (i). Then, there must exist a $\hat{t} < \bar{t}$ such that $\lambda_2(t)$ crosses the line $\frac{4}{3}t$ from the above at the point $\hat{t}$. So $\lambda_2'(\hat{t}) < \frac{4}{3}$. From equation (9), we have $(\lambda_1(\hat{t}) - \hat{t})\lambda_2'(\hat{t}) = \frac{4}{3}\hat{t}$. So we obtain $\frac{\lambda_1(\hat{t})}{\hat{t}} > 2$, which is a contradiction to the fact that $\lambda_1(t)$ lies low the line $2t$ until $\bar{t}$. So this case is also impossible. To sum up the step A to C, we show that $2t > \lambda_1(t)$.

D. It is easy to obtain $\lambda_1(t) > \lambda_2(t)$ and their monotonicity from the differential equation system (8) and (9).

E. Suppose $\lambda_1(t)$ lies entirely under $2t$ and crosses $\frac{4}{3}t$ from above. Then $\lambda_2'(t)$ must rise above $\frac{4}{3}$ first and $\lambda_2(t)$ must cross $\frac{4}{3}t$ from above at least once due to relation (i). Let the crossing happen at $\bar{t}$. So $\lambda_2'(\bar{t}) < \frac{4}{3}$ and $\lambda_2(\bar{t}) = \frac{4}{3}\bar{t}$. From equation (9), we have $(\lambda_1(\bar{t}) - t)\lambda_2'(\bar{t}) = \frac{4}{3}\bar{t}$. Then we can obtain $\frac{\lambda_1(\bar{t})}{\bar{t}} > 2$, which contradicts the fact that $\lambda_1(t)$ lies entirely under $2t$. So $\lambda_1(t)$ must lie above the line $\frac{3}{4}t$.

F. Suppose $\lambda_2(t)$ crosses $\frac{4}{3}t$ at least once. Since $2t > \lambda_1(t)$, there exits a neighborhood around zero such that $\lambda_1'(t) < 2$ for all t belong to this neighborhood. Then $\lambda_2''(t) > 0$ in this neighborhood from relation (i). Suppose the inf of the set of all crossing points of $\frac{4}{3}t$ is $\bar{t}$, where $\bar{t} > 0$ and $\lambda_2'(\bar{t}) < \frac{4}{3}$. From equation (9), we have $(\lambda_1(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \frac{4}{3}\bar{t}$. So we obtain $\frac{\lambda_1(\bar{t})}{\bar{t}} > 2$, which is a contradiction to our obtained conclusion at the end of step C that $2t > \lambda_1(t)$. Then $\lambda_2(t)$ must lie entirely above $\frac{4}{3}t$.

G. Actually, the essence of the above argument can be used to show that $\lambda_2''(t) > 0$. Suppose not, then there must exist a line αt $(2 > \alpha > \frac{3}{4})$ from the origin cutting $\lambda_2(t)$ from below at such a $\bar{t}$, where $\bar{t} > 0$ and $\lambda_2'(\bar{t}) < \alpha$. Again from equation (9), we have $(\lambda_1(\bar{t}) - \bar{t})\lambda_2'(\bar{t}) = \alpha\bar{t}$. So we obtain $\frac{\lambda_1(\bar{t})}{\bar{t}} > 2$, which is a contradiction.

H. Similarly, suppose $\lambda_1''(t) < 0$ does not hold, then there must exist a line αt $(2 > \alpha > \frac{3}{4})$ from the origin cutting $\lambda_2(t)$ from above at such a $\bar{t}$, where $\bar{t} > 0$ and $\lambda_1'(\bar{t}) > \alpha$. From equation (8), we have $(\lambda_2(\bar{t}) - \bar{t})\lambda_1'(\bar{t}) = \alpha\bar{t} - \lambda_2(\bar{t})$, which gives us $\frac{\lambda_2(\bar{t})}{\bar{t}} < \frac{4}{3}$. This is a contradiction. So $\lambda_1''(t) > 0$.

I. Since $\lambda_1''(t) > 0$ and $\lambda_2''(t) < 0$, the result that $\frac{t}{t^*}$ separating $\lambda_1(t)$ and $\lambda_2(t)$ is immediate.

J. To sum up all the above steps, we prove property b) and c).

As to Property e), by replacing $f(v)$ with 1 and $F(v)$ with v in the general formula of $\beta(v)$ derived in the proof of Proposition 2, we obtain the unique solution:

$\beta(v) = \frac{1}{v}\int_0^v (t - \delta t + \delta\beta_2(t))dt = \frac{v}{2}(1-\delta) + \frac{\delta}{v}\int_0^v \beta_2(t)\,dt$. Using the fact that $\frac{3}{4}v > \beta_2(v)$, it is easy to see that $\beta(v)$ is smaller than $\frac{1}{2}v$, where $\frac{1}{2}v$ is the equilibrium bid function for a single auction.

Finally, we show Property d). Its upper bound is shown as follows. At the valuation 1, the second-stage equilibrium payoff for the bidder is $1-t^*$. Given bidders bid truthfully in the first stage, it is necessary for the second-stage bid function to prevent any second-stage deviation. Let the second-stage payoff $\pi = (1-b)\lambda_2(b)$, where b is the choice of the bid. Then $\pi \leq 1-t^*$ for all $b \in [0,1]$. Notice that $\pi > (1-b)\frac{4}{3}b$ because of the fact that $\lambda_2(b) > \frac{4}{3}b$ and $\max(1-b)\frac{4}{3}b = \frac{1}{3}$ when $b = \frac{1}{2}$, so we must have $\frac{1}{3} \leq 1-t^*$, i.e. $t^* \leq \frac{2}{3}$. The lower bound of t^* will be derived in the step B in the proof of Lemma 2 later. Q.E.D.

Proof of Proposition 4.

A. We start with the second-stage bid functions by assuming both bidders follow the same pure strategy bid function at the first stage. We will use bidder 1 to denote the generic bidder for our derivation of the equilibrium bid functions. If bidder 1 wins at the valuation v, then she randomly chooses a bid b on the support $(t_*, t^*]$ with p.d.f. $g^v(b)$ and c.d.f. $G^v(b)$. Her rival will bid according to $\beta_2^v(\hat{v})$, where $\hat{v}$ is the first-stage loser's valuation and the winner believes that $\hat{v}$ is in $[0,v)$ with density $\frac{1}{v}$. The optimality of the randomization requires $(v-b)\Pr(b > \beta_2^v(\hat{v}))$ to be a constant K^v for all $b \in (r_*, r^*)$. Let $\lambda_2^v(\hat{v}) = \beta_2^{v^{-1}}(\hat{v})$. We have $\Pr(b > \beta_2^v(\hat{v})) = \frac{\beta_2^{v^{-1}}(b)}{v} = \frac{\lambda_2^v(b)}{v}$ and $(v-b)\Pr(b > \beta_2^v(\hat{v})) = (v-b)\frac{\lambda_2^v(b)}{v}$. Hence we need that:

(10) $(v-b)\frac{\lambda_2^v(b)}{v} = K^v$

The optimality of bidder 1's rival's strategy requires that: $\beta_2^v(\hat{v}) = \arg\max_{\hat{b}} \left(\hat{v} - \hat{b}\right)\Pr\left(\hat{b} > b\right)$ for all $\hat{v} \in [0, v)$. Let $t = \beta_2^v(\hat{v})$. Since $\Pr(\hat{b} > b) = F^v(t)$, the above expression can be rewritten as $t = \arg\max\,(\hat{v} - t)\, G^v(t)$ for all $\hat{v} \in [0, v)$. The first-order condition leads to:

(11) $\left(\lambda_2^v(t) - t\right) f^v(t) = F^v(t)$

It is almost the same asymmetric auction case examined by Vickrey (1961). It is easy to check that both bidders should have the same ending (maximum) bid. Let the ending bid be t^*. From equation (10), we obtain $\lambda_2^v(b) = \dfrac{vK^v}{v-b}$. By symmetry, this implies that the bid function for bidder 1's rival is $\beta_2^v(\hat{v}) = v - \dfrac{vK^v}{\hat{v}}$. Notice that $\beta_2^v(K^v) = 0$. So in order to obtain a monotonic bid function, we assume that when $\hat{v} \in [0, K^v)$, bidder 1's rival will not bid at all. Similarly, $\lambda_2^v(t^*) = v$, so $\dfrac{vK^v}{v-t^*} = v$, then $t^* = v - K^v$. We adopt the stability refinement argument by Vickrey (1961) and choose the particular equilibrium where $\lambda_2^v(b) = \dfrac{vK^v}{v-b}$ is tangent to the 45 degree line. This implies that $K^v = \dfrac{v}{4}$. So $t^* = \dfrac{3v}{4}$ and $\lambda_2^v(b) = \dfrac{v^2}{4(v-b)}$ and $\beta_2^v(\hat{v}) = v - \dfrac{v^2}{4\hat{v}}$. Substitute the functional form of $\lambda_2^v(.)$ into equation (11), we have $\dfrac{g^v(t)}{G^v(t)} = \dfrac{4(v-t)}{(v-2t)^2}$. Integration on both sides leads to: $\ln G^v(t) = -\ln(2t - v) - \dfrac{v}{2t-v} + C$. For all values of C, we have $G^v(t) \to 0$ when $t \to \dfrac{v}{2}$. So $\dfrac{v}{2}$ is the lower bound of the support of the randomized bids. Also $G^v(t^*) = 1$, i.e., $\ln G^v(t^*) = 0$, so $-\ln(2 \times \dfrac{3v}{4} - v) - \dfrac{v}{2 \times \dfrac{3v}{4} - v} + C = 0$. Then $C = \ln v + 2 - \ln 2$. So $G^v(t) = \dfrac{v}{2(2t-v)} e^{\frac{4t-3v}{2t-v}}$. We have shown that when $\hat{v} \in [0, K^v)$, bidder 1's rival will not bid. Since we find that $K^v = \dfrac{v}{4}$ and bidder 1 will randomize on the support $(\dfrac{v}{2}, \dfrac{3v}{4}]$, we can see that bidder 1's rival will not win at all when $\hat{v} \in [\,K^v, \dfrac{v}{2})$ either. With the bidder symmetry, by now we have found the second-stage bid functions.

B. Now we start to derive the first-stage bid function. Given a valuation v, bidder 1 bids $\beta(z)$ where $z > v$. So we first consider the situation when bidder 1 mimics a higher type and the other bidder truthfully follows $\beta(.)$. Then she wins with probability z and loses with probability $1-z$. If bidder 1 wins, at the second stage, she believes that her rival has the valuation $\hat{v}$ distributed on $[0,z)$ with density $\frac{1}{z}$. Her rival believes that she faces a first-stage winner with valuation z, hence if i) $\hat{v}\in[0,\frac{z}{4})$, then she will not submit a bid. ii) if $\hat{v}\in[\frac{z}{4},z)$, then she will bid according to the bid function $\beta_2^z(\hat{v})=z-\frac{z^2}{4\hat{v}}$. Then the second-stage best response for the first-stage winner given she mimics type z will be to submit a bid b satisfying the following conditions. If $b=0$, then her second-stage payoff is $\frac{v}{4}$. If she submits a nonzero bid, her payoff is $(v-b)\Pr\left(b>\beta_2^z(\hat{v})\right)$, which equals $\frac{(v-b)z}{4(z-b)}$. This term is maximized by choosing $b=0$. In this case, her second-stage best payoff is again $\frac{v}{4}$. So we can see that under all instances, if bidder 1 mimics z at the first stage and wins, she can get $\frac{v}{4}$ at most in the second stage given the other bidder follows the specified equilibrium strategy. Now consider what if bidder 1 loses the first stage by mimicking type z. She knows exactly her rival's valuation $\hat{v}$, where $v<z<\hat{v}$. Her rival will randomly choose a bid b from the interval $(\frac{\hat{v}}{2},\frac{3\hat{v}}{4}]$ according to the c.d.f. $G^{\hat{v}}(b)=\frac{\hat{v}}{2(2b-\hat{v})}e^{\frac{4b-3\hat{v}}{2b-\hat{v}}}$. Then the best response for the bidder will be i) if $v\in[0,\frac{\hat{v}}{4})$, then she will not submit a bid. ii) if $v\in[\frac{\hat{v}}{4},\hat{v})$, then she will bid according to the bid function $\beta_2^{\hat{v}}(v)=\hat{v}-\frac{\hat{v}^2}{4v}$. Next we need to determine the appropriate integration regions for payoff functions under different values of v. First, if $v<\frac{z}{2}$, she will obtain zero second-stage payoff when she loses the first stage according to the stated strategies. Then we can be sure that she never needs to mimic such a z that $v<\frac{z}{2}$ because a) it does not bring her any higher second-stage payoff following winning the first stage due to the fact that a bidder can always obtain $\frac{v}{4}$ at the second stage given winning the first stage no matter what z she mimics and b) it does not bring her any higher second-stage payoff following losing the first stage either since other mimicking type can always bring her nonnegative second-stage payoff. Second, when $\frac{1}{2}>v>\frac{z}{2}$, she will not bid when $\hat{v}\in(2v,1]$ and bid $\hat{v}-\frac{\hat{v}^2}{4v}$ when $\hat{v}\in(z,2v]$. So her second-stage payoff when

$\hat{v} \in (z, 2v]$ is $[v - (\hat{v} - \frac{\hat{v}^2}{4v})] \Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b)$. $\Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b) = F^{\hat{v}}(\hat{v} - \frac{\hat{v}^2}{4v}) = \frac{v}{2v - \hat{v}} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}$. Then $[v - (\hat{v} - \frac{\hat{v}^2}{4v})] \Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b) = \frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}$. So her overall expected second-stage payoff following she losing the first stage is $\pi_1(v, z) = \int_z^{2v} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}} \times \frac{1}{1 - z}) d\hat{v}$. Finally, in the case where $\frac{1}{2} < v$, her overall expected second-stage payoff following losing the first stage is $\pi_z(v, z) = \int_z^1 (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}} \times \frac{1}{1 - z}) d\hat{v}$. Notice that if we can find a first-stage bid function to implement truthful bidding, i.e. $z = v$, then at $v = \frac{1}{2}$, we will have $\pi_1(\frac{1}{2}) = \pi_2(\frac{1}{2})$ and $\pi_1'(\frac{1}{2}) = \pi_2'(\frac{1}{2})$ according to the above derived expressions of $\pi_1(.)$ and $\pi_2(.)$, which ensures the continuity of the first-stage bid function at $v = \frac{1}{2}$. Now we are ready to derive the first-stage bid function. The bidder's first-stage payoff if she observes a valuation v while mimics $z > v$ when the other bidder follows the equilibrium bid functions is:

i) $v \leq \frac{1}{2}$

$\Pi_1(v, z)$

$= z[(v - \beta(z)) + \delta \frac{v}{4}] + (1 - z)[\delta \pi_1(v, z)]$

$= z[(v - \beta(z)) + \delta \frac{v}{4}] + (1 - z)[\delta \int_z^{2v} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}} \times \frac{1}{1 - z}) d\hat{v}]$

$= z[(v - \beta(z)) + \delta \frac{v}{4}] + \delta \int_z^{2v} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}) d\hat{v}$

$= z[(v - \beta(z)) + \delta \frac{v}{4}] - \delta \int_{2v}^{z} (\frac{2v - \hat{v}}{4} e^{\frac{2v - 2\hat{v}}{2v - \hat{v}}}) d\hat{v}$

The first-order condition w.r.t. z yields:

(12) $v - \beta(z) - z\beta'(z) + \frac{1}{4}\delta v - \delta \frac{2v - z}{4} e^{\frac{2v - 2z}{2v - z}} = 0$

In equilibrium $z = v$, so we can have:

(13) $v - \beta(v) - v\beta'(v) = 0$

The unique solution is $\beta(v) = \frac{1}{2}v$.

ii) $v > \frac{1}{2}$

$\Pi_1(v,z) = z[(v - \beta(z)) + \delta\frac{v}{4}] + (1-z)[\delta\pi_2(v,z)]$ and we can obtain the same bid function. The above is the solution when the bidder mimics a type bigger than her own valuation. Also, we need to check when the bidder mimics a lower type whether the bid function still remains the same. It is easy to check that it is indeed the case. Q.E.D.

Proof of Proposition 5.

Let $I_1(0)$ denote the inf of the randomized bids for type zero at the first stage where the subscript represents the number of the stage. We know that $I_1(v) > I_1(0)$ for all $v > 0$ from the monotonicity requirement. Once the zero type bids $I_1(0)$ in the first stage, she will lose with probability one and her zero valuation can be inferred with probability one too. Under this contingency, we have a subsequent second-stage auction where type zero competes with the type uniformly distributed over $[0, 1]$. In this subgame, suppose first that the zero type randomizes over zero. Let $S_2(0) > 0$ denote the sup of her randomized bids at the second stage. We claim that this zero type must be defeated with probability one. Otherwise, the zero type will win with positive probability bringing her negative payoff, which is impossible in equilibrium. Therefore, all her rival's bids must be above $S_2(0)$. However, this will not be optimal for those of her rivals whose types fall in the interval $[0, S_2(0)]$ since these types will obtain negative payoff with probability one in this particular subgame. Hence we obtain that the zero type can not randomizes above zero at the second stage. Then suppose $S_2(0) < 0$, the zero type's rivals can always win with negative bids, which is impossible for any normal auction rule. So finally we consider the only case left where $S_2(0) = 0$. In this case, we can argue that the zero type must be defeated with probability one as before. Then any of type zero's rivals who has a valuation $v > 0$ must bid above zero. But this makes the optimal bid non-existent because whenever the bidder bids $b > 0$, it is always better for her to bid $\frac{b}{2}$. So no equilibrium can exist in the subgame. Then we can conclude that there is no symmetric monotonic equilibrium for the whole two-stage first-price auctions with the announcement of the first-stage losing bid game. Q.E.D.

Proof of Lemma 1.

The equilibrium will be derived in the following step A and B, which at the same time yields the necessary equilibrium condition.

A. To support the second-stage equilibrium, we need to show that $k \leq \frac{1}{2}$. The loser definitely does not want to deviate. We need to further show that the winner does not want to decrease the bid. If the winner decrease the bid by $\varepsilon > 0$, her second-stage payoff will be $\pi = (\hat{v} - ((1-k)v + k\hat{v} - \varepsilon))\frac{(1-k)v + k\hat{v} - \varepsilon - v}{(1-k)v + k\hat{v} - v}$, where $\frac{(1-k)v + k\hat{v} - \varepsilon - v}{(1-k)v + k\hat{v} - v}$ is the probability of winning. We need $\frac{d\pi}{d\varepsilon} \leq 0$. So we have $(\hat{v} - v)(2k-1) \leq 2\varepsilon$. Since this inequality needs to hold no matter how small ε is, we then need $k \leq \frac{1}{2}$.

B. To support the first-stage equilibrium, we need to show that $k \geq \frac{\delta}{1+\delta}$. To show this, we have to derive the first-stage bid functions first. Let us assume bidder 1 observes a valuation v while mimics $z < v$. Then her first-stage payoff is $z(v - \beta(z))$. Conditional on bidder 1 wins the first stage, her rival will have valuation t uniform over $[0, z)$ with density $\frac{1}{z}$ and randomize over $[t,$ $(1-k)t + kz)$. It is easy to check that the best response for bidder 1 will be to bid $(1-k)t + kz$ to win the object for sure. Then her payoff is $\int_0^z [v - ((1-k)t + kz)]\frac{1}{z}dt = v - \frac{z}{2} - \frac{kz}{2}$. Conditional on bidder 1 loses the first stage, her rival will have valuation t uniform over $[z, 1)$ with density $\frac{1}{1-z}$ and bid $(1-k)z + kt$. Then bidder 1's best response will be to bid $(1-k)z + kt + \varepsilon$ as long as $(1-k)z + kt < v$. The deviation of z will be analyzed in the following two categories.

First, we consider a z deviation such that $(1-k)z + kt > v$ when $t = 1$, i.e., $z \in (\frac{v-k}{1-k}, v)$, which represents a small deviation as bidder 1's rival may bid above v for some realizations of

t. We assume bidder 1 can win the object only with $(1-k)z+kt$ at the second stage. If we can obtain an equilibrium under this assumption, the equilibrium will still remain valid under such assumption as bidder 1 needs to win with $(1-k)z+kt+\varepsilon$. Notice that bidder 1 will only bid when $(1-k)z+kt<v$, which implies that $t<\frac{v-(1-k)z}{k}$. So her second-stage payoff is $\int_z^{\frac{v-(1-k)z}{k}}[v-((1-k)z+kt)]\frac{1}{1-z}dt=\frac{(v-z)^2}{2k}\frac{1}{1-z}$.

Then her overall two-stage expected payoff is:

$$\pi=z(v-\beta(z))+\delta z(v-\frac{z}{2}-\frac{kz}{2})+\delta(1-z)\frac{(v-z)^2}{2k(1-z)}$$
$$=zv-z\beta(z)+\delta(zv-\frac{z^2}{2}-\frac{kz^2}{2})+\delta\frac{(v-z)^2}{2k}$$

Differentiate π w.r.t. z and set the first-order condition to zero, we obtain:

(14) $v-z\beta'(z)-\beta(z)+\delta(v-z-kz)-\frac{\delta(v-z)}{k}=0$

In equilibrium, $z=v$, so we have $v-v\beta'(v)-\beta(v)-\delta kv=0$. With usual boundary condition $\beta(0)=0$, we have $\beta(v)=\frac{(1-\delta k)v}{2}$.

Second, we then require the above derived bid functions can also prevent a large z deviation such that $(1-k)z+kt\le v$ when $t=1$, i.e., $z\in[0,\frac{v-k}{1-k}]$ where bidder 1's rival will always bid under or equal to v. Following the above specified strategy and truthful bidding, a bidder's total payoff is $\frac{v^2}{2}+\frac{\delta v^2}{2}$, whose derivation is in the proof of Lemma 4. We next will show that bidder 1 will not deviate to zero at the first stage, which requires that the deviation profit $\delta\int_0^1(v-kt)dt\le\frac{v^2}{2}+\frac{\delta v^2}{2}$. This inequality can be rearranged as $(v-\frac{\delta}{1+\delta})^2-(\frac{\delta}{1+\delta})^2+\frac{\delta k}{1+\delta}\ge 0$ for all v including $v=\frac{\delta}{1+\delta}$. Hence we must have $k\ge\frac{\delta}{1+\delta}$. We then need to show that under the condition $k\ge\frac{\delta}{1+\delta}$, all other big deviations can be prevented too. The total deviation profit is: $\pi=(zv-\frac{(1-\delta k)z}{2})+(\delta zv-\frac{z^2}{2}-\frac{kz^2}{2})+\delta(1-z)\int_z^1(v-(1-k)z-kt)\frac{1}{1-z}dt$. Setting $\frac{d\pi}{dz}=0$, we find the best z deviation is $\frac{v+\delta k-\delta}{1+\delta k-\delta}$. While, we can check that this best deviation fails the large deviation constraint $(1-k)z+k<v$. This means that π is monotonic in z for all $z\in[0,$

$\frac{v-k}{1-k}$]. It is straight forward to check that when $z=\frac{v-k}{1-k}$, the deviation profit π is smaller than the profit from truthful bidding. Therefore all large deviations can be prevented. Similarly, we can analyze the situation where bidder 1 mimics a type $z>v$ in the first stage, which yields the same equilibrium bid functions and equilibrium conditions. So we can conclude that the necessary condition for the above derived bid functions to be an equilibrium is $\frac{1}{2}\geq k\geq\frac{\delta}{1+\delta}$. Q.E.D.

Proof of Proposition 6.

It is straight forward to check that the derived bid functions in Lemma 1 constitute an equilibrium. Q.E.D.

Proof of Lemma 2.

The proof consists of the following three steps.

A. We need to find an appropriate expression for the revenue. The expected payment for one bidder in the first stage is $\int_0^1[\frac{v^2}{2}(1-\delta)+\delta\int_0^v\beta_2(t)\,dt]dv$. The second-stage expected sum of payment can be rearranged as $\int_0^1[\beta_1(v)\beta_2^{-1}(\beta_1(v))+\beta_2(v)\beta_1^{-1}(\beta_2(v))-v\beta_2(v)]\}dv$. So the total revenue is:

$$R_3=2\{\int_0^1\frac{v^2}{2}(1-\delta)\,dv+\int_0^1\delta[\int_0^v\beta_2(t)\,dt+\beta_1(v)\beta_2^{-1}(\beta_1(v))+\beta_2(v)\beta_1^{-1}(\beta_2(v))-v\beta_2(v)]dv\}.$$

Let $P=\int_0^1[\int_0^v\beta_2(t)\,dt+\beta_1(v)\beta_2^{-1}(\beta_1(v))+\beta_2(v)\beta_1^{-1}(\beta_2(v))-v\beta_2(v)]dv$.

Since $\int_0^1\int_0^v\beta_2(t)\,dtdv=\int_0^1(1-v)\beta_2(v)\,dv$, P can then be rewritten as:

$$P=\int_0^1\{\beta_2(v)+\beta_1(v)\lambda_2((\beta_1(v))+\beta_2(v)\lambda_1(\beta_2(v))-2v\beta_2(v)\}dv$$

B. We need to find a bound for P. As there is no closed form solution to the bid functions, our approach is to use their bounds to bound P. We claim that $P<1-\frac{5}{3}t^*+t^{*2}-\frac{1}{3}t^{*4}+\frac{1}{3}t^{*5}$ and $t^*>\frac{5}{8}$, where t^* is the common end point of two second-stage bid functions. To show this, we will first obtain some auxiliary results. Here we abbreviate the terminal bid t^* as t.

Result 1.

$\int_0^t s\lambda_2(s)\lambda_2'(s)\,ds = s\lambda_2(s)\lambda_2(s)|_0^t - \int_0^t \lambda_2(s)(\lambda_2(s) + s\lambda_2'(s))\,ds$

$= t - \int_0^t \lambda_2(s)\lambda_2(s)\,ds - \int_0^t s\lambda_2(s)\lambda_2'(s)\,ds$

So $2\int_0^t s\lambda_2(s)\lambda_2'(s)\,ds = t - \int_0^t [\lambda_2(s)]^2 ds$

Result 2.

Multiply s to both sides of equation (9) and integrate, we have

$\int_0^t s\lambda_1(s)\lambda_2'(s)\,ds - \int_0^t s^2\lambda_2'(s)\,ds = \int_0^t s\lambda_2(s)\,ds$, which leads to:

$\int_0^t s\lambda_1(s)\lambda_2'(s)\,ds - [s^2\lambda_2(s)|_0^t - \int_0^t 2s\lambda_2(s)\,ds] = \int_0^t s\lambda_2(s)\,ds$

So $\int_0^t s\lambda_1(s)\lambda_2'(s)\,ds = t^2 - \int_0^t s\lambda_2(s)\,ds$

Result 3.

$\int_0^t \lambda_2(s)\lambda_1'(s)\,ds$

$= \lambda_2(s)\lambda_1(s)|_0^t - \int_0^t \lambda_2'(s)\lambda_1(s)\,ds = 1 - \int_0^t (\lambda_2(s) + s\lambda_2'(s))\,ds$

$= 1 - [\int_0^t \lambda_2(s)\,ds + \int_0^t s\lambda_2'(s)\,ds] = 1 - [\int_0^t \lambda_2(s)\,ds + s\lambda_2(s)|_0^t - \int_0^t \lambda_2(s)\,ds] = 1 - t$

It is easy to see that $\int_0^t \lambda_1(s)\lambda_2'(s)\,ds = t$.

Result 4.

$\lambda_1(s)\lambda_2'(s) + \lambda_2(s)\lambda_1'(s) = \lambda_1(s) + s\lambda_1'(s) + s\lambda_2'(s)$ by adding equation (6) and (7).

So $\int_0^t [\lambda_1(s) + s\lambda_1'(s) + s\lambda_2'(s)]ds = 1$

$\int_0^t \lambda_1(s)\,ds + s\lambda_1(s)|_0^t - \int_0^t \lambda_1(s)\,ds + s\lambda_2(s)|_0^t - \int_0^t \lambda_2(s)\,ds = 1$

Then $\int_0^t \lambda_2(s)\,ds = 2t - 1$

Result 5.

$\int_0^t s\lambda_2'(s)\,ds = s\lambda_2(s)|_0^t - \int_0^t \lambda_2(s)\,ds = t - \int_0^t \lambda_2(s)\,ds = 1 - t$.

Using the above results, we can find the new expression and the bound for P as follows.

I. $\int_0^1 \beta_2(v)\,dv = \int_0^t s\lambda_2'(s)\,ds = 1 - t$. Then $1 - t < \int_0^1 \frac{3}{4} v\,dv$ because $\beta_2(v) < \frac{3}{4}$, leading to $t > \frac{5}{8}$.

II. $\int_0^1 \beta_1(v)\lambda_2((\beta_1(v))dv < \int_0^1 tv^2 dv = \frac{t}{3}$ because $\beta_1(v) < tv$ and $\lambda_2(\beta_1(v)) < v$.

III. $\int_0^1 [\beta_2(v)\lambda_1(\beta_2(v)) - 2v\beta_2(v)]dv = \int_0^t s\lambda_1(s)\lambda_2'(s)ds - \int_0^t 2s\lambda_2(s)\lambda_2'(s)ds$

$= t^2 - \int_0^t s\lambda_2(s)ds - t + \int_0^t [\lambda_2(s)]^2 ds < t^2 - t + \int_0^t ts(ts-s)ds = t^2 - t + \frac{1}{3}t^5 - \frac{1}{3}t^4$

IV. Therefore, $P < 1 - t + \frac{t}{3} + t^2 - t + \frac{1}{3}t^5 - \frac{1}{3}t^4$

C. We are ready to find the final bound. $P - \int_0^1 \frac{v^2}{2}dv < \frac{5}{6} - \frac{5}{3}t^* + t^{*2} - \frac{1}{3}t^{*4} + \frac{1}{3}t^{*5}$. The expression on the right hand side of the inequality is monotonically decreasing for $t^* \in [\frac{1}{2}, \frac{3}{4}]$. Using the fact that $t^* > \frac{5}{8}$, we obtain a upper bound of $P - \int_0^1 \frac{v^2}{2}dv$ as $\frac{8}{49}$. Since $R_3 = \frac{1}{3} + 2\delta(P - \int_0^1 \frac{v^2}{2}dv)$, we then have $R_3 < \frac{1}{3} + \frac{16}{49}\delta$. Q.E.D.

Proof of Lemma 3.

We need to show that the overall revenue has the following expression:

$$R_4 = 2 \times \{\int_0^1 \frac{v^2}{2}(1+\delta)dv + \delta[\int_0^{\frac{1}{2}}(e^2 \int_0^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt)dv + \int_{\frac{1}{2}}^1 (e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt)dv]\}$$

Once this result can be obtained, we can immediately reach the conclusion that $R_4 > \frac{1}{3} + \frac{1}{3}\delta$ because $2 \times \int_0^1 \frac{v^2}{2}(1+\delta)dv = \frac{1}{3} + \frac{1}{3}\delta$ and $\int_0^{\frac{1}{2}}(e^2 \int_0^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt)dv + \int_{\frac{1}{2}}^1 (e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt)dv > 0$. Now we start to derive the expression of R_4. At the valuation v, if bidder 1 loses the first stage and the winner's valuation is inferred as $\hat{v}$, where $\hat{v}$ is uniformly distributed in $(v, 1]$ with density $\frac{1}{1-v}$, then she will bid $\beta_2^{\hat{v}}(v) = \hat{v} - \frac{\hat{v}^2}{4v}$. Her rival will randomize over $(\frac{\hat{v}}{2}, \frac{3\hat{v}}{4}]$ with c.d.f. $G^v(b) = \frac{\hat{v}}{2(2b-\hat{v})} e^{\frac{4b-3\hat{v}}{2b-\hat{v}}}$. If $v > \frac{1}{2}$, the probability for bidder 1 to win the second stage is $\Pr(\hat{v} - \frac{\hat{v}^2}{4v} > b) = G^v(\hat{v} - \frac{\hat{v}^2}{4v}) = \frac{v}{2v-\hat{v}} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}}$. Her expected payment is $\int_v^1 \frac{4v\hat{v} - \hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v})d\hat{v}$. Similarly if $v \leq \frac{1}{2}$, her expected payment is $\int_v^{2v} \frac{4v\hat{v} - \hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v})d\hat{v}$. Next consider the case where bidder 1 wins the first stage at the valuation v. Then her rival will have valuation $\hat{v}$ distributed over the support $[0, v)$ with density

$\frac{1}{v}$. The probability for bidder 1 to win the second stage is $\Pr(b > v - \frac{v^2}{4\hat{v}}) = \Pr(\hat{v} < \frac{v^2}{4(v-b)}) = \frac{v}{4(v-b)}$. So the expected payment for bidder 1 following her winning the first stage is: $\int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}})$. We use $T_{v>\frac{1}{2}}$ and $T_{v<\frac{1}{2}}$ to denote the overall expected payments for bidder 1 at the second stage when $v > \frac{1}{2}$ and $v < \frac{1}{2}$ respectively. Then we have:

$$T_{v>\frac{1}{2}} = v\int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}) + (1-v)\int_v^1 \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v}) d\hat{v}$$
$$= v\int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}) + \int_v^1 \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} d\hat{v}$$

$T_{v>\frac{1}{2}}$ can be rearranged as:

$$T_{v>\frac{1}{2}} = \frac{e^2}{8} v^3 \int_0^{\frac{v}{2}} \frac{v+t}{t^3} e^{-\frac{v}{t}} dt + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt$$
$$= \frac{e^2}{8} v^3 [\int_0^{\frac{v}{2}} \frac{v}{t^3} e^{-\frac{v}{t}} dt + \int_0^{\frac{v}{2}} \frac{1}{t^2} e^{-\frac{v}{t}} dt] + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt$$

Notice that $\int_0^{\frac{v}{2}} \frac{v}{t^3} e^{-\frac{v}{t}} dt = -\frac{1}{v} \int_0^{\frac{v}{2}} -\frac{v}{t} e^{-\frac{v}{t}} d(-\frac{v}{t}) = -\frac{1}{v}(-\frac{v}{t} e^{-\frac{v}{t}} - e^{-\frac{v}{t}})|_0^{\frac{v}{2}} = \frac{3}{v} e^{-2}$ and $\int_0^{\frac{v}{2}} \frac{1}{t^2} e^{-\frac{v}{t}} dt = \frac{1}{v} \int_0^{\frac{v}{2}} e^{-\frac{v}{t}} d(-\frac{v}{t}) = \frac{1}{v}(e^{-\frac{v}{t}})|_0^{\frac{v}{2}} = \frac{1}{v} e^{-2}$. Therefore, $T_{v>\frac{1}{2}} = \frac{v^2}{2} + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt$. With similar method, we can find $T_{v\leq\frac{1}{2}}$ as follows.

$$T_{v\leq\frac{1}{2}} = v\int_{\frac{v}{2}}^{\frac{3v}{4}} b \times \frac{v}{4(v-b)} d(\frac{v}{2(2b-v)} e^{\frac{4b-3v}{2b-v}}) + (1-v)\int_v^{2v} \frac{4v\hat{v}-\hat{v}^2}{4(2v-\hat{v})} e^{\frac{2v-2\hat{v}}{2v-\hat{v}}} (\frac{1}{1-v}) d\hat{v}$$
$$= \frac{v^2}{2} + e^2 \int_0^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt$$

To sum up, the expected payment T for bidder 1 at the second stage is:

$$T = \frac{v^2}{2} + e^2 \int_0^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt \quad \text{when } v \leq \frac{1}{2}$$
$$= \frac{v^2}{2} + e^2 \int_{v-\frac{1}{2}}^{\frac{v}{2}} \frac{v^2-t^2}{t} e^{-\frac{v}{t}} dt \text{ when } v > \frac{1}{2}$$

Then it is straight forward to obtain the desired expression for R_4. Q.E.D.

Proof of Lemma 4.

The expected payment for a bidder with valuation v at the second stage is: $Q_2 = v\int_0^v [(1-k)t + kv] \frac{1}{v} dt = \frac{1}{2} v^2 (1+k)$. The expected payment for a bidder with valuation v

at the first stage is: $Q_1 = v[\frac{(1-\delta k)v}{2}] = \frac{(1-\delta k)v^2}{2}$. So $R_5 = 2 \times \int_0^1 (Q_1 + \delta Q_2) dv = 2 \times \int_0^1 (\frac{(1-\delta k)v^2}{2} + \frac{1}{2}\delta v^2 (1+k)) dv = \frac{1}{3} + \frac{1}{3}\delta$. Q.E.D.

References

[1] Athey, Susan (2001), "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Game of Incomplete Information," *Econometrica*, 69, 851-890.

[2] Blume, Andreas (2003) ,"Bertrand Without Fudge," *Economics Letters*, 78,167-168.

[3] Blume, Andreas (1998) ,"Contract renegotiation with time-varying valuations," *Journal of Economics and Management Strategy*, 7, 397-433.

[4] Kaplan, Todd R. and Shmuel Zamir (2000), "The Strategic Use of Seller Information in Private-Value Auctions," Working Paper, University of Exeter and Hebrew University of Jerusalem, SSRN Electronic Paper Collection, August 21, 2000.

[5] Kennan, John (2001) ,"Repeated Bargaining with Persistent Private Information," *Review of Economic Studies*, 68, 719-755.

[6] Krishna, Vijay (2002), "Auction Theory ," Academic Press.

[7] Landsberger, Michael, Jacob Rubinstein, Elmar Wolfstetter and Shmuel Zamir (2001) ,"First Price Auctions When the Ranking of Valuations is Common Knowledge," *Review of Economic Design*, 6, 461-480.

[8] Maskin, Eric and John Riley(2003), "Uniqueness of Equilibrium in Sealed High Bid Auction," *Games and Economic Behavior*, 45, 395-409.

[9] Milgrom, Paul R. and Robert J. Weber (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089-1122.

[10] Skreta, Vasiliki (2000) ,"Learning the Distribution of Valuations in Second Price Auctions," Working paper, University of Minnesota.

[11] Van den Berg, Gerard J., Jan C. van Ours and Menno P. Pradhan (2001) ,"Declining Price Anomaly in Dutch Rose Auctions," *American Economic Review*, 91, 1055-1062.

[12] Vickrey, William (1961), “Counterspeculation, Auctions, and Competitive Sealed Tenders,” *Journal of Finance*, 16, 8-30.

[13] Weber, Robert J. (1983) , “Multiple Object Auctions,” in R. Engelbrecht-Wiggans, M. Shubik, and R. Stark (eds.), *Auctions, Bidding and Contracting: Use and Theory*, New York University Press, 165-191.